PPV review wksht #1

– Make Key –

*no calculator

1. * Find the area of one petal of the rose curve given by $r = 3cos3\theta$.
2. *Find the area of the region common to the two regions bounded by $r = -6cos\theta \;\; and \; r = 2 - 2cos\theta$.
3. *Find the length of the arc from $\theta = 0 \; to \; \theta = 2\pi$ for the cardioid $r = 2 - 2cos\theta$.

   use $\frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$

4. *Find the intersections of $r = 1 - 2cos\theta \;\; and \; r = 1$.
5. * Find the horizontal and vertical tangent lines of $r = sin\theta \quad 0 \le \theta \le \pi$.

   $\cos 2\theta = \cos^2\theta - \sin^2\theta$
   $\sin 2\theta = 2\sin\theta\cos\theta$

6. * Find the equation of the tangent line to the curve at the given parametric value.
   $x = 4cos\theta \quad and \quad y = 3sin\theta \qquad \theta = \frac{3\pi}{4}$
7. Find the arc length $x = t^2 \;\; y = 4t^3 - 1 \qquad t \in [-1,1]$ — Larson 10.2/30
8. *Find all the points (if any) of horizontal and vertical tangency to the curve — Larson 10.2/21

$$x = 1 - t \quad y = t^3 - 3t$$

9. *Find the velocity and acceleration vectors if the position vector $r(t) =< \sin(3t)\,, \cos(5t) >$
10. *A particle moves in an elliptical path so that its position at any time $t \ge 0$ is given by

$$r(t) = (4sint)i + (2cost)j$$

   a) Find the velocity and acceleration vectors.
   b) Find the velocity, acceleration and speed at $t = \frac{\pi}{4}$.

11. A particle moves in the plane with velocity vector $v(t) =< t - 3\pi cos\pi t, 2t - \pi sin\pi t >$ at t=0, the particle is at the point (1,5)
    a) *Find the position of the particle at t=4.
    b) What is the total distance traveled by the particle from t=0 to t=4

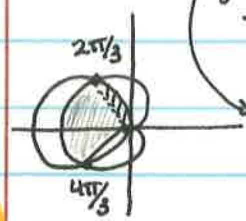
PPV #1

$u = 6\theta$
$du = 6\,d\theta$

① $\frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2 d\theta = \frac{1}{2}\int_{0}^{\pi/6} 2(9\cos^2 3\theta)\,d\theta = \frac{9}{2}\int_0^{\pi/6} 1+\cos 6\theta\; d\theta$

$\frac{9}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{9}{2}\left[\frac{\pi}{6} + \frac{1}{6}\sin\pi\right] = \frac{3\pi}{4}$

ex 3 p 700 Larson

② 

symmetry

The pen shaded area lies between ther circle and the line $\theta = \frac{2\pi}{3}$ b/c the circle is @ $(0, \pi/2)$ at the pole you can

$\frac{A}{2} = \frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2 d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (2-2\cos\theta)^2 d\theta$

then add the pencil shaded area lies btwn $\theta = 2\pi/3$ an $\theta = \pi$ and the cardiod

$= 18\int_{\pi/2}^{2\pi/3} \cos^2\theta\, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta)\,d\theta$

$= 9\int_{\pi/2}^{2\pi/3} (1+\cos 2\theta)\,d\theta + \int_{2\pi/3}^{\pi} 3 - 4\cos\theta + \cos 2\theta\; d\theta$

$= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi}$

$9\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2}\right) + \left(3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right) = \frac{5\pi}{2} = \frac{A}{2} \quad \therefore A = 5\pi$

③ Ex 4 p 701 Larson

$r = 2 - 2\cos\theta \qquad r' = 2\sin\theta$

$\int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2}\; d\theta = \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\; d\theta$

$= \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\; d\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\; d\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\; d\theta$

remember $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$4\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = -8\cos\frac{\theta}{2}\Big|_0^{2\pi} =$

16

(4) $r = 1 - 2\cos\theta = 1$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

(5) HT & VT lines $r = \sin\theta \quad 0 \le \theta \le \pi$

ex 5 p691

write eqns in parametric form.

$x = r\cos\theta = \sin\theta\cos\theta \qquad y = r\sin\theta = \sin\theta\sin\theta = \sin^2\theta$

$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = \cos 2\theta = 0 \qquad 2\theta = \frac{\pi}{2} \qquad 2\theta = \frac{3\pi}{2}$

$\theta = \frac{\pi}{4} \qquad \theta = \frac{3\pi}{4}$

$\frac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \qquad 2\theta = \pi \qquad 2\theta = 0$

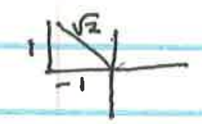
$\theta = \frac{\pi}{2} \qquad \theta = 0$

$(r,\theta) \qquad (r,\theta)$

$\therefore$ VT @ $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) \quad \left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

HT @ $(0,0) \quad (1, \pi/2)$

(6) $x = 4\cos\theta \qquad y = 3\sin\theta \qquad @\ \theta = \frac{3\pi}{4}$



$\frac{dx}{d\theta} = -4\sin\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = \frac{3\cos\theta}{-4\sin\theta} \quad @\ \theta = \frac{3\pi}{4} \quad \frac{3\cos\frac{3\pi}{4}}{-4\sin 3\pi/4} = \frac{-3\left(\frac{\sqrt{2}}{2}\right)}{-4\left(\frac{\sqrt{2}}{2}\right)} = \frac{3}{4} = m$

$x = 4\left(\frac{-\sqrt{2}}{2}\right) = -2\sqrt{2} \qquad y = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} \qquad \left(-2\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$

$y - \frac{3\sqrt{2}}{2} = \frac{3}{4}(x + 2\sqrt{2})$

(7)

#30 p684 10.2

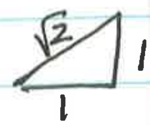
8 Larson 10.2/21

9. $r(t) = \langle \sin(3t), \cos(5t) \rangle$

$v(t) = r'(t) = \langle 3\cos 3t, -5\sin(5t) \rangle$

$a(t) = v'(t) = \langle -9\sin 3t, -25\cos 5t \rangle$

10. $r(t) = (4\sin t)i + (2\cos t)j$



$v(t) = r'(t) = (4\cos t)i + (-2\sin t)j$

$a(t) = v'(t) = (-4\sin t)i + (-2\cos t)j$

$v(\frac{\pi}{4}) = 4\cos\frac{\pi}{4} i - 2\sin\frac{\pi}{4} j = 4(\frac{\sqrt{2}}{2})i - 2(\frac{\sqrt{2}}{2})j = \boxed{2\sqrt{2}i - \sqrt{2}j}$

$a(\frac{\pi}{4}) = (-4\frac{\sqrt{2}}{2})i + (-2\frac{\sqrt{2}}{2})j = \boxed{-2\sqrt{2}i - \sqrt{2}j}$

11. $v(t) = \langle t - 3\pi\cos\pi t, 2t - \pi\sin\pi t \rangle$ @$t=0$ $\langle 1,5 \rangle$

$s(t) = \int v(t)$

$\int (t - 3\pi\cos\pi t)i + (2t - \pi\sin\pi t)j$

$s(t) = (\frac{t^2}{2} - 3\sin\pi t)i + (t^2 + \cos\pi t)j + C$

$s(0) = i + 5j = 0i + j + C$ $\quad C = i + 4j$

$s(t) = (\frac{t^2}{2} - 3\sin\pi t + 1)i + (t^2 + \cos\pi t + 4)j$

$s(4) = (8+1)i + (16+1+4)j = \langle 9, 21 \rangle$

b) $\int_0^4 \sqrt{(t - 3\pi\cos\pi t)^2 + (2t - \pi\sin\pi t)^2}$

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EXAMPLE 3 Finding the Area of a Region Between Two Curves

Find the area of the region common to the two regions bounded by the following curves.

$r = -6\cos\theta$ Circle

$r = 2 - 2\cos\theta$ Cardioid

Solution Because both curves are symmetric with respect to the x-axis, you can work with the upper half-plane, as shown in Figure 10.37. The gray shaded region lies between the circle and the radial line $\theta = 2\pi/3$. Because the circle has coordinates $(0, \pi/2)$ at the pole, you can integrate between $\pi/2$ and $2\pi/3$ to obtain the area of this region. The region that is shaded red lies between the radial lines $\theta = 2\pi/3$ and $\theta = \pi$ and the cardioid. Thus, you can find the area of this second region by integrating between $2\pi/3$ and π. The sum of these two integrals gives the area of the common region lying *above* the polar axis.

$$\frac{A}{2} = \overbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3}(-6\cos\theta)^2\,d\theta}^{\text{Region between circle and radial line } \theta = 2\pi/3} + \overbrace{\frac{1}{2}\int_{2\pi/3}^{\pi}(2 - 2\cos\theta)^2\,d\theta}^{\text{Region between cardioid and radial line } \theta = 2\pi/3 \text{ and } \theta = \pi}$$

$$= 18\int_{\pi/2}^{2\pi/3}\cos^2\theta\,d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi}(4 - 8\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= 9\int_{\pi/2}^{2\pi/3}(1 + \cos 2\theta)\,d\theta + \int_{2\pi/3}^{\pi}(3 - 4\cos\theta + \cos 2\theta)\,d\theta$$

$$= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi}$$

$$= 9\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2}\right) + \left(3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{5\pi}{2}$$

Finally, multiplying by 2, you conclude that the total area is 5π.

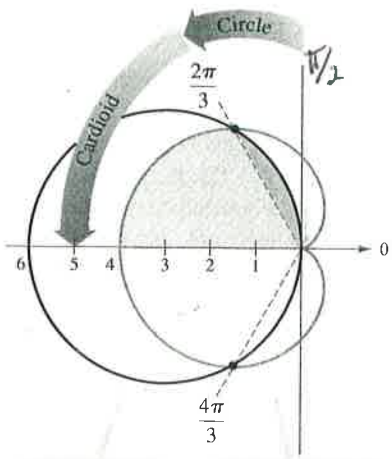

FIGURE 10.37

REMARK To check the reasonableness of the result obtained in Example 3, note that the area of the circular region is $\pi r^2 = 9\pi$. Thus, it seems reasonable that the area of the region lying inside the circle and the cardioid is 5π.

To see the benefit of using polar coordinates for finding the area in Example 3, consider the following integral, which gives the comparable area in rectangular coordinates.

$$\frac{A}{2} = \int_{-4}^{-3/2}\sqrt{2\sqrt{1 - 2x} - x^2 - 2x + 2}\,dx + \int_{-3/2}^{0}\sqrt{-x^2 - 6x}\,dx$$

Try using a computer and numerical integration to show that you obtain the same area as found in Example 3.

Arc Length in Polar Form

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations. (See Exercise 61.)

REMARK When applying the arc length formula to a polar curve, be sure that the curve is traced out only once on the interval of integration. For instance, the rose given by $r = \cos 3\theta$ is traced out once on the interval $0 \leq \theta \leq \pi$, but is traced out twice on the interval $0 \leq \theta \leq 2\pi$.

THEOREM 10.8 Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

#3

EXAMPLE 4 Finding the Length of a Polar Curve

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid

$$r = f(\theta) = 2 - 2\cos\theta$$

as shown in Figure 10.38.

Solution Because $f'(\theta) = 2\sin\theta$, you can find the arc length as follows.

$$\begin{aligned} s &= \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta && \text{Formula for arc length} \\ &= \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta \\ &= 2\sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta \\ &= 2\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta \\ &= 4\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta && \sin\frac{\theta}{2} \geq 0 \text{ for } 0 \leq \theta \leq 2\pi \\ &= -8\cos\frac{\theta}{2}\Big]_0^{2\pi} \\ &= 8 + 8 \\ &= 16 \end{aligned}$$

In the fifth step of the solution, it is legitimate to write

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\sin(\theta/2)$$

rather than $\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\,|\sin(\theta/2)|$ because $\sin(\theta/2) \geq 0$ for $0 \leq \theta \leq 2\pi$.

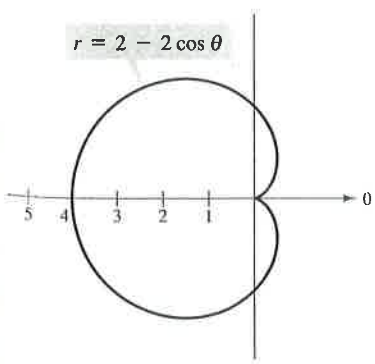

FIGURE 10.38
The arc length of this cardioid is 16.

REMARK Using Figure 10.38, you can determine the reasonableness of this answer by comparing it with the circumference of a circle. For example, a circle of radius $\frac{5}{2}$ has a circumference of $5\pi \approx 15.7$.

Slope and Tangent Lines

To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To convert to polar form, use the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Using the parametric form of dy/dx given in Theorem 10.1, you have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

which establishes the following theorem.

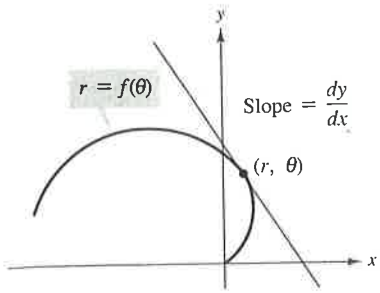

FIGURE 10.27
Tangent line to polar curve.

THEOREM 10.5 Slope in Polar Form

If f is a differentiable function of θ, then the *slope* of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ). (See Figure 10.27.)

From Theorem 10.5, you can make the following observations.

1. Solutions to $\frac{dy}{d\theta} = 0$ yield horizontal tangents, provided that $\frac{dx}{d\theta} \neq 0$.
2. Solutions to $\frac{dx}{d\theta} = 0$ yield vertical tangents, provided that $\frac{dy}{d\theta} \neq 0$.

If $dy/d\theta$ and $dx/d\theta$ are *simultaneously* 0, then no conclusion can be drawn about tangent lines.

#5

EXAMPLE 5 Finding Horizontal and Vertical Tangent Lines

Find the horizontal and vertical tangent lines of $r = \sin\theta$, $0 \leq \theta \leq \pi$.

Solution Begin by writing the equation in parametric form.

$$x = r\cos\theta = \sin\theta\cos\theta$$

and

$$y = r\sin\theta = \sin\theta\sin\theta = \sin^2\theta$$

Next, differentiate x and y with respect to θ and set each derivative equal to 0.

$$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = \cos 2\theta = 0 \quad \longrightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta = 0 \quad \longrightarrow \quad \theta = 0, \frac{\pi}{2}$$

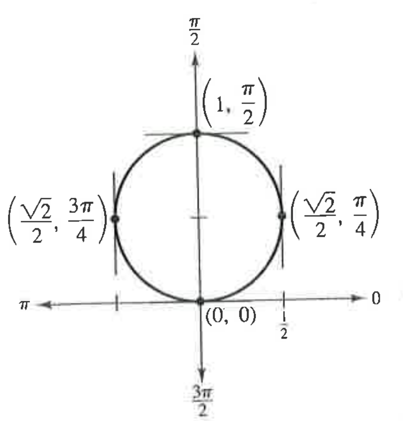

FIGURE 10.28
Horizontal and vertical tangent lines of $r = \sin\theta$.

Thus, the graph has vertical tangent lines at $(\sqrt{2}/2, \pi/4)$ and $(\sqrt{2}/2, 3\pi/4)$, and it has horizontal tangent lines at $(0, 0)$ and $(1, \pi/2)$, as shown in Figure 10.28.

(r, θ)

27. $x = \sec\theta, \quad y = \tan\theta$

Horizontal tangents:

$$\frac{dy}{d\theta} = \sec^2\theta \neq 0; \text{ none}$$

Vertical tangents:

$$\frac{dx}{d\theta} = \sec\theta\tan\theta = 0 \text{ when } \theta = 0, \ \pi.$$

Points: $(1, 0), (-1, 0)$

28. $x = \cos^2\theta, \quad y = \cos\theta$

Horizontal tangents:

$$\frac{dy}{d\theta} = -\sin\theta = 0 \text{ when } \theta = 0, \ \pi.$$

Since $\dfrac{dx}{d\theta} = 0$ at these values, exclude them.

Vertical tangents:

$$\frac{dx}{d\theta} = -2\cos\theta\sin\theta = 0 \text{ when } \theta = \frac{\pi}{2}, \ \frac{3\pi}{2}.$$

(Exclude $0, \ \pi$.)

Point: $(0, 0)$

29. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad 0 \le t \le \frac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t), \quad \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{2e^{-2t}}\, dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\, dt = \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$$

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30. $x = t^2, \quad y = 4t^3 - 1, \quad -1 \le t \le 1, \quad \dfrac{dx}{dt} = 2t, \quad \dfrac{dy}{dt} = 12t^2$

$$s = \int_{-1}^{1} \sqrt{4t^2 + 144t^4}\, dt = 2\int_0^1 2t\sqrt{1 + 36t^2}\, dt = \frac{1}{18}\int_0^1 (1 + 36t^2)^{1/2}(72t)\, dt = \left[\frac{1}{27}(1 + 36t^2)^{3/2}\right]_0^1 \approx 8.30$$

31. $x = t^2, \quad y = 2t, \quad 0 \le t \le 2$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2, \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$s = 2\int_0^2 \sqrt{t^2 + 1}\, dt = \left[t\sqrt{t^2 + 1} + \ln\left|t + \sqrt{t^2 + 1}\right|\right]_0^2 = 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916$$

32. $x = \arcsin t, \quad y = \ln\sqrt{1 - t^2}, \quad 0 \le t \le \frac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \quad \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = -\frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1 - t^2)^2}}\, dt = \int_0^{1/2} \frac{1}{1 - t^2}\, dt = \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2} = -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$